

New potentials for conformal mechanics

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Abstract

We find under some mild assumptions that the most general potential of 1-dimensional conformal systems with time independent couplings is expressed as $V = V_0 + V_1$, where V_0 is a homogeneous function with respect to a homothetic motion in configuration space and V_1 is determined from an equation with source a homothetic potential. Such systems admit at most an $SL(2, \mathbb{R})$ conformal symmetry which, depending on the couplings, is embedded in $\text{Diff}(\mathbb{R})$ in three different ways. In one case, $SL(2, \mathbb{R})$ is also embedded in $\text{Diff}(S^1)$. Examples of such models include those with potential $V = \alpha x^2 + \beta x^{-2}$ for arbitrary couplings α and β , the Calogero models with harmonic oscillator couplings and non-linear models with suitable metrics and potentials. In addition, we give the conditions on the couplings for a class of gauge theories to admit a $SL(2, \mathbb{R})$ conformal symmetry. We present examples of such systems with general gauge groups and global symmetries that include the isometries of $AdS_2 \times S^3$ and $AdS_2 \times S^3 \times S^3$ which arise as backgrounds in AdS_2/CFT_1 .

1 Introduction

It has been known for sometime that 1-dimensional models with potential $V = \beta x^{-2}$ are conformally invariant [1, 2]. de Alfaro, Fubini and Furlan (DFF) explored the $SL(2, \mathbb{R})$ conformal symmetry of this theory and noticed that the Hamiltonian operator does not have a ground state [2]. To overcome this problem, they suggested to choose the eigenstates of

$$\mathcal{O} = \frac{p^2}{2} + \alpha x^2 + \beta x^{-2} , \quad (1.1)$$

as a basis in the Hilbert space. \mathcal{O} is not the Hamiltonian operator, but a linear combination of conserved charges associated with the $SL(2, \mathbb{R})$ conformal symmetry of the theory. Choosing suitably the coupling constants α, β this operator exhibits a ground state and discrete energy spectrum. As a result the DFF formulation of the theory has been widely accepted in the literature. However, although a Hilbert space has been defined for the theory, the Hamiltonian operator is not diagonal in the chosen basis and so the energy levels of the theory cannot be identified. There have been many generalizations of the $V = \beta x^{-2}$ model, see eg [3]-[8], including the construction of non-linear theories¹ [9, 10], which exhibit similar properties, see also reviews [11, 12] and references within. The DFF treatment of the theory and its generalizations have found widespread applications in the description of near horizon black hole dynamics [13, 14, 15, 16] and in the understanding of black hole moduli spaces [17, 18, 19, 20, 21].

Another application of conformal mechanics is in the context of AdS_2/CFT_1 correspondence [22], and for further exploration see eg [23, 24]. It is expected that string theory or M-theory on a $AdS_2 \times X$ background is dual to a conformal theory on the boundary. After analytic continuation the Lorentzian boundary of AdS_2 , which is two copies of \mathbb{R} , is mapped to a circle, see eg [23]. In the Euclidean regime, the associated dual theory should be a conformal theory defined on the circle. As we shall demonstrate, there are such conformal theories but they are based on different potentials² from $V = -\beta x^{-2}$.

In this paper, we investigate the conformal properties of theories with Lagrangian

$$L = \frac{1}{2} g_{ij} \dot{q}^i \dot{q}^j - V , \quad (1.2)$$

where g is a metric on the configurations space, V is a potential and \dot{q} is the time derivative of the position. The conditions required for such theories to be invariant under the conformal transformations (2.1) have been stated in (2.2). Assuming that the configuration space of these theories admits a homothetic vector field Z associated with a homothetic potential h , the conditions for conformal invariance (2.2) can be solved. The potential of the theory can be written as

$$V = V_0 + V_1 , \quad (1.3)$$

¹With the term “linear theories” we mean those for which the configuration space is \mathbb{R}^n equipped with the Euclidean metric but they may exhibit a non-trivial potential. “Non-linear theories” are those with curved configuration space.

²The $SL(2, \mathbb{R})$ conformal symmetry of the $V = -\beta x^{-2}$ model acts with fractional linear time reparameterizations and it cannot be embedded in $\text{Diff}(S^1)$. We allow for diffeomorphisms with some discontinuities.

where V_0 is a homogeneous function with respect to the homothetic motion Z and V_1 obeys the inhomogeneous equation (2.13) which has source the homothetic potential h . The dimension of the conformal group of these models is at most three and one of the generators is time translations. This is because the parameter of the transformation obeys a third order equation (2.10). The maximal conformal group is $SL(2, \mathbb{R})$ and it is embedded in $\text{Diff}(\mathbb{R})$ in three different ways generating the vector fields

$$\begin{aligned} (i) \quad & \partial_t, \quad t\partial_t, \quad t^2\partial_t; \\ (ii) \quad & \partial_t, \quad \cosh(\omega t)\partial_t, \quad \sinh(\omega t)\partial_t; \\ (iii) \quad & \partial_t, \quad \cos(\omega t)\partial_t, \quad \sin(\omega t)\partial_t; \end{aligned} \tag{1.4}$$

for some ω related to the couplings.

The first $SL(2, \mathbb{R})$ embedding (i) in (1.4) is realized for the models with $V_1 = 0$. These class of models has a homogeneous potential V_0 and includes the DFF model, and its linear and non-linear generalizations [9, 10]. Furthermore, if $V_1 \neq 0$, the $SL(2, \mathbb{R})$ conformal group is embedded in $\text{Diff}(\mathbb{R})$ generating the vector fields (ii) or (iii). These are Newton-Hooke transformations and the two cases are distinguished by the sign of the inhomogeneous term in the equation (2.13) which determines V_1 . The models with conformal transformation (ii) and (iii) are related by a naive analytic continuation, and the $SL(2, \mathbb{R})$ group in the latter case can be embedded in $\text{Diff}(S^1)$.

The class of conformal models with conformal symmetry (ii) and (iii) in (1.4) includes those with potential [3]

$$V = \alpha x^2 + \beta x^{-2}, \tag{1.5}$$

where $V_0 = \beta x^{-2}$ and $V_1 = \alpha x^2$. For $\alpha < 0$ the conformal group generates the vector fields (ii) in (1.4), while for $\alpha > 0$ the conformal group generates the vector field (iii). There are also several multi-particle models which exhibit type (ii) and (iii) in (1.4) conformal symmetry. Such systems include the Calogero model with harmonic oscillator couplings of equal frequency [27], and the multi-particle linear models of [28] for which V_0 satisfies additional symmetries. We shall present some additional linear and non-linear systems with (ii) and (iii) conformal symmetries. Observe that the theories with $\alpha, \beta > 0$ in (1.5) have a ground state and discrete energy spectrum, and so there is no need to choose another operator different from the Hamiltonian to give a basis in the Hilbert space of the theory. This also applies to several other models in this class.

One result which follows from the general analysis of this paper is that the most general linear conformal model admits a potential (1.3), where V_0 is a homogeneous function of the positions q of degree -2 and $V_1 = \alpha|q|^2$. This rigidity result is based on the uniqueness of homothetic motions in flat space associated with a homothetic potential. The homothetic motion is the homogeneous scaling of all coordinates, $q^i \rightarrow \ell q^i$. These models admit an $SL(2, \mathbb{R})$ conformal symmetry generated by the vector fields (ii) and (iii) in (1.4) and depending on whether $\alpha < 0$ or $\alpha > 0$, respectively.

More recently, conformal models in one dimension have been investigated which apart from scalar fields contain also vectors [25]. So far such theories have been based on gauging models with homogeneous potentials. We shall demonstrate that such models can be generalized to include potentials of the type (1.5). In particular, we derive the

conditions (4.9) for gauged non-linear sigma models with Lagrangian (4.1) to admit a conformal symmetry, and determine the equations that restrict the potentials. We find that for a large class of such conformal theories the potential can be written as in (1.3), where both V_0 and V_1 must also be gauge invariant. In addition, we give some examples which include conformal models with a general gauged group and global symmetries. Some of these models exhibit the isometries of $AdS_2 \times S^3$ and $AdS_2 \times S^3 \times S^3$ backgrounds as global symmetries. A class of these models is solvable, and the Hamiltonian has a ground state and discrete spectrum. A similar investigation of $SL(2, \mathbb{R})$ symmetries in the context of matrix models has been done in [26] and the associated potentials have been identified.

This paper is organized as follows. In section 2, we derive investigate the conditions for conformal invariance of non-linear 1-dimensional theories and derive the scalar potential (1.3). In section 3, we give several examples of such models. In section 4, we derive the conditions on the couplings gauged sigma models with a potential to admit conformal invariance, and give several examples. In section 5, we present our conclusions.

2 Conformal models

2.1 Lagrangian

Consider the Lagrangian (1.2) of a sigma model on a manifold M with metric g and with a potential V . This describes either the propagation of a non-relativistic particle in a curved manifold M or a multi-particle system with a non-trivial configuration space M . One can assign mass dimensions such that q is dimensionless $[q] = 0$ while $[t] = -1$. Thus $[L] = 2$ provided one takes the coupling V terms to have dimension 2. This is not the most general Lagrangian that one can consider as a coupling with dimension 1 has not been included. This will be done elsewhere [29].

2.2 Conformal transformations

All time re-parameterizations $t' = u(t)$ are conformal transformations of the Euclidean metric on \mathbb{R} as $ds^2 = (dt')^2 = (\dot{u})^2 dt^2$. Therefore, one can choose any of these transformations and demand that leave the action (1.2) invariant. Apart from time translations³, such transformations will not leave the action invariant unless there is a compensating additional transformation on the positions generated by a vector field X on M [9]. As a result, one considers the infinitesimal transformations [10]

$$\delta q^i = -\epsilon a(t) \dot{q}^i + \epsilon X^i(t, q) , \quad (2.1)$$

where ϵ is a small parameter. The first term in the transformation of q is induced by the infinitesimal transformation $\delta t = \epsilon a(t)$, where $a(t)$ is the vector field on \mathbb{R} which generates the time re-parameterizations, while the second term containing X is the compensating transformation which may explicitly depend on t .

³We have chosen the couplings g and V not to depend explicitly on time. However, it is straightforward to carry out the analysis of this section for models with time-dependent couplings.

The conditions for the invariance of the action (1.2), up to surface terms, under the transformations (2.1) are [10]

$$\mathcal{L}_X g_{ij} = \dot{a} g_{ij} , \quad \partial_t X^i g_{ij} = \partial_i f , \quad \dot{a} V + X^k \partial_k V = -\partial_t f \quad (2.2)$$

where $f = f(t, q)$ is the contribution from the surface term, and where ∂_t denotes differentiation of the explicit dependence of X and f on t , ie

$$\frac{d}{dt} f(q, t) = \partial_t f + \dot{q}^i \partial_i f . \quad (2.3)$$

The conserved charges associated with the above symmetries are

$$Q(a, X) = \frac{a}{2} g_{ij} \dot{q}^i \dot{q}^j - g_{ij} \dot{q}^i X^j + a V + f . \quad (2.4)$$

It can be easily shown that $Q(a, X)$ is conserved subject to field equations.

2.3 Solution of conformal conditions and new models

It is clear that the first condition in (2.2) implies that X generates a family of homothetic transformations on M which may depend on t . Since all $\text{Diff}(\mathbb{R})$ are conformal transformations, the system can be invariant under any subgroup of $\text{Diff}(\mathbb{R})$. So, one should consider at most as many homothetic motions in M as the dimension of the subgroup of conformal transformations. However, in most examples of interest M admits one homothetic motion generated by a vector field Z which does not depend explicitly on t

$$\mathcal{L}_Z g_{ij} = \ell g_{ij} , \quad (2.5)$$

where ℓ is a constant. Then, the first condition can be solved by setting

$$X^i(t, q) = \ell^{-1} \dot{a}(t) Z^i(q) . \quad (2.6)$$

Assuming that Z arises from a homothetic potential, ie

$$Z^i g_{ij} = \partial_j h , \quad (2.7)$$

where $h = h(q)$, f can be chosen⁴

$$f = \ell^{-1} \ddot{a} h . \quad (2.8)$$

The last equation in (2.2) can now be rewritten as

$$\dot{a}(V + \ell^{-1} Z^k \partial_k V) = -\ell^{-1} \partial_t^3 a h . \quad (2.9)$$

⁴We assume that $\ddot{a} \neq 0$. If $\ddot{a} = 0$, Z does not have to be associated with a homothetic potential and V is a homogeneous function of the homothetic motion. The models do not have a $SL(2, \mathbb{R})$ symmetry but rather are invariant under time translations and scale transformations generated by the vector fields $\partial_t, t\partial_t$.

Since we are seeking to find potentials V which solve the above equations and do not depend explicitly on t , we have to take

$$\partial_t^3 a = \lambda \dot{a} , \quad (2.10)$$

where λ is a constant. Of course, if $\dot{a} = 0$, there is no condition on V as the only symmetry of the action is time translations. Thus, we take $\dot{a} \neq 0$ and as a result the equation which determines the potential is

$$V + \ell^{-1} Z^k \partial_k V = -\ell^{-1} \lambda h . \quad (2.11)$$

The general solution for the potential can be written as in (1.3), ie $V = V_0 + V_1$, where V_0 is the most general solution of the homogenous equation

$$V_0 + \ell^{-1} Z^k \partial_k V_0 = 0 , \quad (2.12)$$

and V_1 is a solution of

$$V_1 + \ell^{-1} Z^k \partial_k V_1 = -\ell^{-1} \lambda h . \quad (2.13)$$

Clearly, there are 3 cases to consider depending on whether $\lambda = 0$, or $\lambda > 0$ or $\lambda < 0$. In these three choices, the vector field a is determined from (2.10) as follows. For $\lambda = 0$, one has

$$a = a_0 + a_1 t + a_2 t^2 , \quad (2.14)$$

where a_0, a_1 and a_2 are integration constants. For $\lambda = \omega^2$, one has

$$a = a_0 + b e^{\omega t} + c e^{-\omega t} , \quad (2.15)$$

and for $\lambda = -\omega^2$, one has

$$a = a_0 + b \cos(\omega t) + c \sin(\omega t) , \quad (2.16)$$

where a_0, b, c are integration constants. The new conformal models arise from the last two cases.

Before we proceed to investigate individual models, let us examine the algebra of these transformations. A basis in the space of vector fields of the infinitesimal transformations (2.14), (2.15) and (2.16) is given in (i), (ii) and (iii) of (1.4), respectively, with $|\lambda| = \omega^2$. The group of transformations generated by (2.14), (2.15) and (2.16) is $SL(2, \mathbb{R})$. However, $SL(2, \mathbb{R})$ is embedded into $\text{Diff}(\mathbb{R})$ in three different ways⁵. The group of transformations generated by (2.16) is also embedded in the $\text{Diff}(S^1)$ as the associated vector fields are periodic in t . The two cases (2.15) and (2.16) are related to each other by analytic continuation.

Substituting the above expressions of X into the conserved charges and using the properties of the homothetic motion on M , one finds that

$$Q(a, Z) = \frac{a}{2} g_{ij} \dot{q}^i \dot{q}^j - \dot{a} \ell^{-1} \partial_i h \dot{q}^i + a(V_0 + V_1) + \ell^{-1} \ddot{a} h . \quad (2.17)$$

These can be easily computed explicitly in the examples described below.

⁵In the (2.14) case, $SL(2, \mathbb{R})$ acts with fractional linear transformations on \mathbb{R} .

3 Examples

3.1 Conformal particle in flat space

The most illuminating model is that of a single particle propagating on the real line. Here we shall show that (1.5), which has been found previously in [3], is the only potential consistent with conformal invariance. For this we shall take the Lagrangian

$$\mathcal{L} = \frac{1}{2}\dot{x}^2 - V(x) , \quad (3.1)$$

and we shall determine V such that the action is conformally invariant. For this consider the homothetic vector field

$$Z = \frac{1}{2}x\partial_x , \quad (3.2)$$

on the configuration space. For this choice of Z , $\ell = 1$. The homothetic potential in this case is

$$h = \frac{1}{4}x^2 . \quad (3.3)$$

Then the equation (2.12) can be solved for V_0 to yield

$$V_0 = \beta x^{-2} , \quad (3.4)$$

for some constant β , which is the potential of the DFF model. However, we have seen that the potential V also receives a contribution from V_1 which is determined in (2.13). The latter equation can be solved as

$$V_1 = \alpha x^2 , \quad \alpha = -\lambda/8 . \quad (3.5)$$

Thus the most general potential $V = V_0 + V_1$ of such conformal models is given in (1.5).

The Hamiltonian of this class of conformal models is given in (1.1). As it has already been mentioned the associated Hamiltonian operator with $\alpha > 0, \beta \geq 0$ has a ground state and discrete spectrum.

3.2 Conformal multi-particle systems

Consider next the linear model of N particles propagating in \mathbb{R} and interacting with a potential V . The Lagrangian of such a system is

$$\mathcal{L} = \frac{1}{2} \sum_i^N (\dot{x}^i)^2 - V(x^i) . \quad (3.6)$$

To find the potentials V consistent with conformal invariance, consider the homothetic motion

$$Z = \frac{1}{2} \sum_{i=1}^N x^i \partial_i , \quad (3.7)$$

of \mathbb{R}^N configuration space. The homothetic potential in this case is

$$h = \frac{|x|^2}{4} , \quad |x|^2 = \delta_{ij} x^i x^j . \quad (3.8)$$

As it has been mentioned in the introduction, Z in (3.7) is the unique homothetic motion in \mathbb{R}^N associated with a homothetic potential⁶ up to an overall scale which does not affect the form of the potential. After solving the conditions (2.12) and (2.13), one finds that the potential V is

$$V = \alpha |x|^2 + V_0(x) , \quad \alpha = -\lambda/8 \quad (3.9)$$

and V_0 is a homogeneous function of degree -2

$$x^i \partial_i V_0 = -2V_0 . \quad (3.10)$$

(3.9) is the most general potential of linear models.

Of course, there are many choices for V_0 . A minimal choice for V_0 is $V_0 = \beta |x|^{-2}$. However, this is not unique. For example, one can also choose

$$V_0 = \sum_{i \neq j} \frac{\beta_{ij}}{(x^i - x^j)^2} . \quad (3.11)$$

The models with potentials V given in (3.9) and (3.11) are the Calogero models with harmonic couplings of equal frequency. Our results demonstrate that these models are conformally invariant. It is well-known that such models with $\alpha > 0$ and $\beta \geq 0$ have a vacuum state and discrete energy spectrum [27, 30]. Of course, there are many more potential functions V_0 which satisfy the homogeneity condition (3.10) above than those appearing in the Calogero models. The above models also include those presented in [28] where some additional symmetry assumptions were made on the form of V_0 potential.

To summarize, we have shown that all the above models admit either an $SL(2, \mathbb{R})$ conformal symmetry which is embedded in $\text{Diff}(\mathbb{R})$ as in (i), (ii) or (iii) of (1.4) depending on whether $\alpha = 0$, $\alpha < 0$ or $\alpha > 0$, respectively. The associated conserved charges can be computed by a direct substitution in (2.17).

3.3 Particles propagating on cones

So far, we have presented linear models as examples. For a non-linear example, consider particles propagating on a cone and interacting with a potential V . The Lagrangian of such a system is

$$\mathcal{L} = \frac{1}{2} (\dot{r}^2 + r^2 \gamma_{ij} \dot{x}^i \dot{x}^j) - V(r, x) , \quad (3.12)$$

⁶If the requirement of the homothetic potential is removed, the scaling transformation (3.7) can mix with other isometries, like $SO(N)$ rotations, to give rise to new homothetic motions. These can be used to construct invariant theories under subgroups of $SL(2, \mathbb{R})$ involving at most two generators.

where γ is the metric of the cone section which does not depend on the radial coordinate r but it may depend on the rest of the coordinates x . The cone metric

$$ds^2 = dr^2 + r^2 \gamma_{ij} dx^i dx^j , \quad (3.13)$$

admits a homothetic motion generated by the vector field

$$Z = \frac{1}{2} r \partial_r , \quad (3.14)$$

which homothetic potential

$$h = \frac{r^2}{4} + k(x) , \quad (3.15)$$

where k is an arbitrary function of x . It is straightforward to show that the most general potential compatible with conformal symmetry is

$$V = \alpha r^2 + \beta(x) r^{-2} + 8\alpha k(x) , \quad \alpha = -\lambda/8 . \quad (3.16)$$

Again these models admit a $SL(2, \mathbb{R})$ conformal symmetry generating the vector fields (i), (ii) or (iii) of (1.4) depending on whether $\alpha = 0$, $\alpha < 0$ or $\alpha > 0$, respectively.

4 Conformal gauge theories in one dimension

4.1 Action

Motivated by applications in *AdS/CFT*, which typically requires dual theories with a gauge symmetry, and to enhance the class of 1-dimensional conformal systems, we shall also examine the conditions for a gauged sigma model to admit conformal invariance. For this, we assume that M admits a group of isometries G , generating the vector fields ξ , which leave V invariant. Gauging the isometries of (1.2), one finds the Lagrangian⁷

$$L = \frac{1}{2} g_{ij} \nabla_t q^i \nabla_t q^j - V , \quad (4.1)$$

where

$$\nabla_t q^i = \dot{q}^i - A^a \xi_a^i , \quad [\xi_a, \xi_b] = -f_{ab}^c \xi_c , \quad (4.2)$$

A is the gauge potential and f are the structure constants of G . We assign mass dimension to A as $[A] = 1$ so that L has mass dimension 2.

The equations of motion of the theory are

$$g_{ij} D_t \nabla_t q^j + \partial_i V = 0 , \quad \xi_{ia} \nabla_t q^i = 0 , \quad (4.3)$$

⁷This is not the most general Lagrangian of dimension 2 as couplings of dimension 1 have not been included.

where

$$D_t \nabla_t q^i = \partial_t \nabla_t q^i - A^a \partial_j \xi_a^i \nabla_t q^j + \Gamma_{jk}^i \nabla_t q^j \nabla_t q^k . \quad (4.4)$$

Under certain conditions the gauge connection A can be eliminated from the equations of motion leading to a theory with dynamical variables just the q 's. In particular notice that the second equation of motion can be rewritten as

$$\ell_{ab} A^b = \xi_{ia} \dot{q}^i \quad (4.5)$$

where $\ell_{ab} = g_{ij} \xi_a^i \xi_b^j$. If ℓ is invertible, then all A can be eliminated. However, we shall not elaborate on this here. Instead, we shall proceed to find the conditions such that the action (4.1) is invariant under some conformal symmetries.

4.2 Conformal and gauge symmetries

The action (4.1) is invariant under the gauge transformations

$$\delta q^i = \eta^a \xi_a^i , \quad \delta A^a = \nabla_t \eta^a , \quad (4.6)$$

where η is the gauge infinitesimal parameter.

Next as in the un-gauged case, one expects that the transformations on q and A , which induce the conformal symmetries of the action (4.1), to contain two parts. One part is associated with time re-parameterizations and an additional term which generates compensating transformations on the configuration space. As a result, we postulate the conformal transformations

$$\begin{aligned} \delta q^i &= -\epsilon a(t) \partial_t q^i + \epsilon X^i(t, q, A) , \\ \delta A^a &= -\epsilon \dot{a} A^a - \epsilon a \dot{A}^a + \epsilon W^a(t, q, A) , \end{aligned} \quad (4.7)$$

where the first term in the variation of q and the first two terms in the variation of A are the transformations induced on q and A from the infinitesimal re-parameterization of t , $\delta t = \epsilon a(t)$, and the rest are the compensating transformations.

These transformation mix with the gauge transformations above. In particular, the coordinate transformation induced on A by a can be rewritten as a gauge transformation with parameter $-a A^a$. Since the action is invariant under gauge transformations, this can be used to simplify the conformal transformations as

$$\begin{aligned} \delta q^i &= -\epsilon a(t) \nabla_t q^i + \epsilon X^i , \\ \delta A^a &= \epsilon W^a . \end{aligned} \quad (4.8)$$

For the same reason X and Z are not uniquely defined. In particular X and W are defined up to terms $\ell^a \xi_a$ and $\nabla_t \ell^a$, respectively, where $\ell = \ell(t, q, A)$.

Assuming that X and W do not depend on time derivatives of q , a straightforward computation reveals that the conditions required for the invariance of the action, up to surface terms, are

$$\begin{aligned} \mathcal{L}_X g_{ij} &= \dot{a} g_{ij} , \\ g_{ij} \partial_t X^j + g_{ij} A^a [\xi_a, X]^j - g_{ij} \xi_b^j W^b &= \partial_i f , \\ \dot{a} V + X^k \partial_k V &= -\partial_t f , \end{aligned} \quad (4.9)$$

where $f = f(t, q)$ is the contribution from the surface term. f is taken to be gauge invariant, $\xi_a^i \partial_i f = 0$. To find conformal models, one has to solve (4.9).

4.3 Solution of conformal conditions

Here, we shall not seek the most general solution to the conformal invariance conditions(4.9). Instead, we shall take

$$[\xi_a, X] = 0, \quad W^a = 0. \quad (4.10)$$

In this case, the above conditions (4.9) reduce to those of (2.2) but with the additional assumption that f is gauge invariant.

To find solutions, we proceed as in section 2.3. The potential is given as $V = V_0 + V_1$, (1.3), with V_0 and V_1 determined by the equations (2.12) and (2.13), respectively. There is an additional restriction here that the homothetic potential h is gauge invariant, $\xi_a^i \partial_i h = 0$.

As in the systems without gauge symmetry, there are three cases to consider depending on whether $\lambda = 0$, $\lambda > 0$ or $\lambda < 0$. In all cases the conformal group is $SL(2, \mathbb{R})$ but it is embedded in three different ways into $\text{Diff}(\mathbb{R})$. The $\lambda > 0$ and $\lambda < 0$ models are related by analytic continuation.

4.4 Examples

4.4.1 Gauged nonlinear models on a cone

Examples of non-linear gauge theories exhibiting conformal symmetry are those that describe the propagation of particles on a cone. Assuming that the cone section metric γ admits a group of isometries generating the vector fields ξ , the Lagrangian of the theory can be written as

$$\mathcal{L} = \frac{1}{2}(\dot{r}^2 + r^2 \gamma_{ij} \nabla_t x^i \nabla_t x^j) - V(r, x), \quad (4.11)$$

where

$$\nabla_t x^i = \dot{x}^i - \xi_a^i A^a. \quad (4.12)$$

The homothetic vector field is again given by $Z = \frac{1}{2} r \partial_r$ and commutes with the Killing vector fields ξ_a satisfying the assumption (4.10).

The rest of the analysis proceed as in the cone example in section 3.3 for the un-gauged model yielding a potential

$$V = \alpha r^2 + \beta(x) r^{-2} + 8\alpha k(x), \quad \alpha = -\lambda/8, \quad (4.13)$$

where now $\beta(x)$ and $k(x)$ are gauge invariant functions of the cone section, $\xi_a^i \partial_i \beta = \xi_a^i \partial_i k = 0$. The simplest explicit example is to consider the flat cone \mathbb{R}^2 and as the gauged symmetry the rotational symmetry. The potential of this model is given as in (4.24) with β and k constants.

4.4.2 Gauge theories

A large class of linear conformal models⁸ can be constructed beginning from some gauge group G and some linear representation D of its Lie algebra \mathfrak{g} on a vector space \mathcal{V} .

⁸These can also be thought of as special cases of the cone models above.

Suppose that D leaves invariant a (constant) metric g on \mathcal{V} . Then one can consider the Lagrangian

$$L = \frac{1}{2} g_{mn} \nabla_t x^m \nabla_t x^n - V(x) , \quad (4.14)$$

where

$$\nabla_t x^m = \dot{x}^m - A^a (D_a)^m{}_n x^n . \quad (4.15)$$

To determine V such that this theory is conformal, observe that the metric admits a homothetic motion generated by the vector field

$$Z = \frac{1}{2} x^m \partial_m . \quad (4.16)$$

Moreover, this commutes with the Killing vector fields

$$\xi_a = \frac{1}{2} (D_a)^m{}_n x^n \partial_m , \quad (4.17)$$

ie $[Z, \xi_a] = 0$. As a consequence (4.10) is satisfied. Furthermore, the homothetic potential of Z is

$$h = \frac{1}{4} g_{mn} x^m x^n . \quad (4.18)$$

Using this, the potential V can be determined by solving (2.12) and (2.13) as

$$V = \alpha g_{mn} x^m x^n + V_0 , \quad \alpha = -\frac{\lambda}{8} , \quad (4.19)$$

and V_0 is a function of x of homogeneous degree -2,

$$x^m \partial_m V_0 = -2V_0 , \quad (4.20)$$

which is also invariant under G . The minimal choice is

$$V_0 = \frac{\beta}{g_{mn} x^m x^n} . \quad (4.21)$$

However such a choice is not unique for general gauge groups and representations D . A similar potential has been derived in the investigation of $SL(2, \mathbb{R})$ invariant matrix models in [26].

Amongst these models, one can take as $D = \mathfrak{adj} \otimes I^k$, where \mathfrak{adj} is the adjoint representation of a group G and I is the trivial representation. In such a case, the Lagrangian can be written as

$$L = \frac{1}{2} g_{ab} \kappa_{ij} \nabla_t x^{ai} \nabla_t x^{bj} - V(x) \quad (4.22)$$

where

$$\nabla_t x^{ai} = \dot{x}^{ai} - A^b f_{bc}{}^a x^{ci} , \quad (4.23)$$

g_{ab} is an invariant metric on the adjoint representation of G and κ a metric on the k -copies of the trivial representation. The potential in this case can be written as

$$V = \alpha g_{ab} \kappa_{ij} x^{ai} x^{bj} + V_0, \quad \alpha = -\frac{\lambda}{8}, \quad (4.24)$$

and V_0 is a function of x of homogeneous degree -2 which is also invariant under G . Now there are several options for V_0 . For example, V_0 can be any homogeneous function of degree -2 expressed in terms of the gauge invariant functions like

$$m^{ij} = g_{ab} x^{ai} x^{bj}, \quad m^{ijk} = f_{abc} x^{ai} x^{bj} x^{ck}, \quad (4.25)$$

and many others which can be constructed from all the invariant tensors of \mathfrak{g} under the action of the adjoint representation. One example is a gauged Calogero model for which the potential is given in (4.24) with

$$V_0 = \sum_{i \neq j} \frac{\beta_{ij}}{g_{ab} (x^{ai} - x^{aj})(x^{bi} - x^{bj})}. \quad (4.26)$$

Further restrictions can be put on the form of the potential by requiring that the theory is invariant under the global symmetry $\times_i O(n_i)$ which leaves κ invariant. The above construction can also be done by replacing $\mathfrak{ad} \mathfrak{j}$ with another representation of the gauge group.

This class of conformal theories has all the bosonic symmetries required for the CFT duals of backgrounds like $AdS_2 \times S^3$ or $AdS_2 \times S^3 \times S^3$. In particular, one can easily construct models with rigid symmetry $SL(2, \mathbb{R}) \times SO(4)$, which is the isometry group of $AdS_2 \times S^3$, and any gauge symmetry including $U(N)$, and similarly there are models which exhibit the isometries of $AdS_2 \times S^3 \times S^3$ backgrounds as symmetries. It is also worth remarking that the analytic continuation of a $\lambda > 0$ theory which exhibits $SL(2, \mathbb{R})$ conformal symmetry is equivalent to taking λ to $-\lambda$ and V_0 to $-V_0$ and leads to a model with $SL(2, \mathbb{R})$ conformal invariance but now embedded in $\text{Diff}(S^1)$ as expected in the context of AdS_2/CFT_1 .

The quantum theory of the model with action (4.14) can be easily described in the case that $V_0 = 0$ and $\alpha > 0$. The Hilbert space of these theories can be constructed starting from the Hilbert space of $\dim D$ harmonic oscillators. Then, gauge invariance requires that one has to consider only those states which are invariant under the gauge group. The Hamiltonian operator has a ground state and the spectrum is discrete. However the details of the construction depend on the choice of gauge group and representation D . If $V_0 \neq 0$, the quantum theory depends on the choice of V_0 . It is likely that some of the properties of the $V_0 = 0$ models can be maintained in the presence of a large class of V_0 potentials as it happens for the Calogero models with harmonic oscillator couplings.

5 Concluding remarks

We have demonstrated that the potential V of conformal mechanics models admitting a homothetic motion in configuration space can be expressed as a sum $V = V_0 + V_1$, where

V_0 is a homogeneous function of the homothetic motion and V_1 is determined from an equation which has as a source the homothetic potential. Depending on the couplings, the maximal conformal group $SL(2, \mathbb{R})$ is embedded in $\text{Diff}(\mathbb{R})$ in three different ways. Furthermore, one of these can also be thought as an embedding of $SL(2, \mathbb{R})$ in $\text{Diff}(S^1)$. This is significant from the point of view of AdS_2/CFT_1 as the dual Euclidean theory must be defined on the boundary which is a circle.

Examples of conformal 1-dimensional systems include models with potential $V = \alpha x^2 + \beta x^{-2}$ [3]. The $SL(2, \mathbb{R})$ conformal symmetry of this model is embedded in $\text{Diff}(\mathbb{R})$ in three different ways depending on whether $\alpha = 0$, $\alpha < 0$ or $\alpha > 0$, respectively. Moreover if $\alpha > 0$, $SL(2, \mathbb{R})$ can also be embedded in $\text{Diff}(S^1)$.

We have described all 1-dimensional linear conformal theories described by the Lagrangian (3.6). The potential of all such models is $V = \alpha|x|^2 + V_0$, where V_0 is a homogeneous of degree -2 function of the positions x . This rigidity result is based on the uniqueness of the homothetic motion in flat space associated with a homothetic potential and the analysis in section 2. Examples of such theories include the Calogero models with harmonic oscillator couplings of equal frequency as well as the models given in [28]. We have also presented examples of non-linear models.

It is clear from the analysis of section 2 that if the configuration space of a system admits a single homothetic motion associated with a homothetic potential, then the vector field $a(t)\partial_t$ which generates the time re-parameterizations obeys the third order equation (2.10). Because of this, the conformal group can be at most 3-dimensional. Therefore, if there are theories with larger conformal groups than $SL(2, \mathbb{R})$, then necessarily must have additional fields, like vectors or spinors, and possibly must couple to gravity. As a consequence all linear models admit at most a $SL(2, \mathbb{R})$ conformal symmetry.

We have also investigated the conformal properties of 1-dimensional systems with scalar and vector fields based on the Lagrangian(4.1). We have derived the conditions for such systems to admit a conformal symmetry (4.9) and present several examples. The potential of a class of such theories is again the sum of a homogeneous function, under the action the homothetic motion, and a term that depends on the homothetic potential. Examples of such conformal models can exhibit general gauge groups and global symmetries. In particular, we have constructed models with arbitrary gauge group which have the isometries of $AdS_2 \times S^3$ and $AdS_2 \times S^3 \times S^3$ backgrounds as global symmetries. Similar potentials have arisen in the investigation of matrix models with $SL(2, \mathbb{R})$ invariance in [26].

Gravitational backgrounds that have applications in AdS_2/CFT_1 typically preserve some of the spacetime supersymmetry and as a result the dual theories must be superconformal. The supersymmetric extension of some of the conformal models we have considered here has already been done, see eg [30] and [9, 10] for the supersymmetric extension of Calogero model with harmonic oscillator couplings and that of non-linear conformal theories with homogeneous potentials, respectively, see also [32] for matrix models. Conformal linear models with extended supersymmetry and homogenous potentials have been reviewed in [12], see also [33]. It is straightforward to construct superconformal models with potentials $V = V_0 + V_1$ specially those that exhibit a small number of supersymmetries. Such supersymmetric extensions can be based on the results of [31, 19] and they will be reported elsewhere.

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